CALCULATION OF THE SELF-INDUCTANCE OF SINGLE-LAYER COILS.

By Edward B. Rosa.

1. THE FORMULÆ.

A well-constructed standard of self-inductance may be measured with considerable precision in terms of a resistance. If its value be computed from its dimensions, the two results should agree, provided the resistance is known in absolute measure. This affords a method of determining resistance in absolute measure, the success of which depends, of course, on how accurately the self-inductance can be computed, as well as upon the accuracy of the measurements.

Kirchhoff's method of determining resistance in absolute measure consists essentially in finding by experiment the mutual inductance of a pair of coils in terms of a certain resistance, the value of the mutual inductance having been calculated from the dimensions of the coils and their distance apart. Placing these two values equal to each other, the value of the given resistance becomes known in absolute measure. The mutual inductance of such a pair of coils can be determined experimentally by two measurements of self-inductance, in one of which the current flows in the same direction in the two coils and in the other it flows in opposite directions. If L and L' are the two values of self-inductance determined experimentally, L_1 and L_2 the self-inductances of the two coils separately, and M their mutual inductance, then

$$L = L_1 + 2M + L_2$$
$$L' = L_1 - 2M + L_2$$
$$\therefore M = \frac{L - L'}{4}.$$

Such a pair of coils may have a large radius and relatively small rectangular cross section, such as used by Rowland and Glazebrook, or they may be coaxial solenoids of equal length, or one may be long and the other quite short, the short one being either within or without the long one. All these cases may be calculated quite accurately when the dimensions are accurately known.

The simplest method, however, is to use a single coil of relatively large dimensions, wound with a single layer of wire, (1) calculating its self-inductance from its dimensions and (2) measuring its self-inductance directly in terms of a resistance. The dimensions of such a coil may be measured with great precision, supposing it wound on an accurately ground marble cylinder. In an article in the last number of this Bulletin Professor Coffin[1] gave a description of such a standard of inductance, and calculated its value by two different formulæ, one being an absolute formula using elliptic integrals, first given by Lorenz,[2] and the second a formula derived by Coffin in the form of a converging series, being an extension of Rayleigh's formula for a cylindrical winding of a single layer. These two formulæ give results agreeing to within 1 part in 50,000, an agreement which is highly satisfactory. As the measurements of the coil may readily be made of sufficient accuracy, the only question remaining is whether the two closely agreeing formulæ are as accurate when applied to the coil in question as they appear to be.

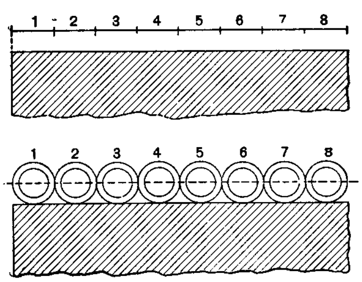

Fig. 1.

In deriving the formulæ the current is supposed to be uniformly distributed over the surface of the cylinder, whereas the coil is actually wound with 661 turns of round wire, having a diameter of 0.0634 cm and insulated by a covering 0.0030 cm thick. Instead of the single-current sheet *ab* first assumed in deriving the formulæ, the current sheet is subsequently supposed to be divided into n sections having altogether a self-inductance n^2 times as great as the single current sheet. This can be realized if we assume a winding of n turns of a flat strip, l/n wide (where l is the length of the wind-

[1] No. 4, March, 1906. [2] Wied. Annalen, 7, p. 170; 1879.

ing), and no thickness, wound uniformly over the cylinder, at a distance of $\frac{l}{2n}$ cm from the surface of the cylinder. If the edges of the strip come together but do not make electrical contact, such a winding would be equivalent to a uniform current sheet and the formulæ for the latter would apply if n^2 is inserted as a factor in L, n being the number of turns. In the case of the coil constructed for the Bureau of Standards, when $n=661$, the wire is so small and the number of turns so large it was assumed by Coffin that the actual winding is substantially equivalent to the current sheet to which the formulæ strictly apply.

2. CASE OF SHORT COIL.

Let us first examine the case of a short coil having a radius of 25 cm and a length of 1 cm, wound with 10 turns of wire, the bare wire being 0.08 cm and the covered wire 0.10 cm in diameter. Coffin's formula for a short single-layer coil of radius a and length b reduces in this case to

$$L=4\pi n^2 a\left\{\left(1+\frac{b^2}{32a^2}\right)\log\frac{8a}{b}+\frac{b^2}{128a^2}-0.5\right\} \qquad (1)$$

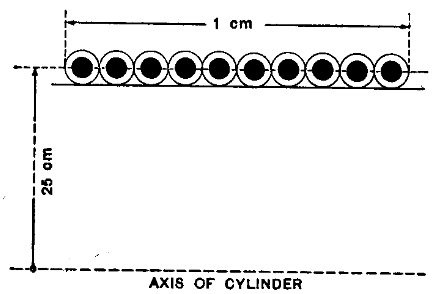

Fig. 2.

the terms neglected amounting to less than one part in a hundred million of L. This is the formula given by Rayleigh[3] in 1881.

Substituting $a=25$, $b=1$, $n=10$, we have for the self-inductance

$$L=4\pi a\times 100\left\{\left(1+\frac{1}{32\times 625}\right)\log_e 200+\frac{1}{128\times 625}-0.5\right\}$$

or, $L=4\pi a\times 479.8595$ cm.

This value assumes the current flowing in a sheet, as though the coil were wound with 10 turns of very thin tape 1 mm wide, the radius of the winding being 25 cm.

[3] Proc. Roy. Soc., **32**, 1881, and Collected Papers.

The summation formula for the self-inductance of a single-layer coil of n turns is as follows:

$$L = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \dots + 2M_{1n} \quad (2)$$

where L_1 is the self-inductance of a single turn, M_{12} is the mutual inductance of any two adjacent turns, M_{13} is the mutual inductance of the first and third or any two turns separated by one, and M_{1n} is the mutual inductance of the first and last turns. For a coil of 10 turns this becomes

$$L = 10L_1 + 18M_{12} + 16M_{13} + 14M_{14} + \dots\dots\dots + 2M_{1\,10} \quad (3)$$

If we calculate L_1 and the nine Ms, we can compute the self-inductance of the actual coil wound with wire of any particular size, the only assumption being that the current is uniformly distributed over the cross section of the wire. Max Wien's formula[4] for the self-inductance of a single circular turn of wire of radius a and radius of cross section ρ is

$$L = 4\pi a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho} - 1.75 - .0083\frac{\rho^2}{a^2}\right\} \quad (4)$$

This term should be: $+\frac{\rho^2}{24a^2}$ (Rayleigh-Niven Formula)

Substituting $a = 25$, $\rho = .04$, we have

$$L_1 = 4\pi a\left\{\left(1+\frac{2}{(2500)^2}\right)\log_e 5000 - 1.75 - .0083\frac{16}{(2500)^2}\right\}$$

$$= 4\pi a \times 6.76720 \text{ cm.}$$

Maxwell's formula for the mutual inductance of two coaxial circles near each other reduces to the following when the two radii are equal, a being the common radius and b the distance between the circles.

$$M = 4\pi a\left\{\left(1+\frac{3}{16}\frac{b^2}{a^2}\right)\log\frac{8a}{b} - 2.0 - \frac{1}{16}\frac{b^2}{a^2}\right\} \quad (5)$$

Hence, $$M_{12} = 4\pi a\left\{\left(1+\frac{3}{16}\frac{.01}{625}\right)\log_e 2000 - 2 - \frac{1}{16}\frac{.01}{625}\right\}$$

$$= 4\pi a \times 5.600924 \text{ cm.}$$

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[4] Wied. Annalen, **53**, p. 934; 1894.

Giving b the successive values 0.2, 0.3, 0.4, etc., the other values of the Ms are derived. The following values of the 10 terms of the summation formula (3) are thus obtained:

$10\ L_1 = 67.6720$	$8\ M_{17} = 30.4779$
$18\ M_{12} = 100.8166$	$6\ M_{18} = 21.9346$
$16\ M_{13} = 78.5254$	$4\ M_{19} = 14.0898$
$14\ M_{14} = 63.0344$	$2\ M_{1\,10} = 6.8099$
$12\ M_{15} = 50.5787$	
$10\ M_{16} = 39.9189$	Sum $= 473.8582 = \dfrac{L}{4\pi a}$

This is less than the value found for $\frac{L}{4\pi a}$ by formula (1) by 6.0013, which is about 1.25 per cent. It will be noticed that the values of the mutual inductances are independent of the size of the wire with which the coil is wound, but that the self-inductance is not. Thus, if the wire were only half a millimeter in diameter, $10\ L_1$ would be 79.3034. In the latter case the total would be 5.6139 *more* by the summation formula than by the current sheet formula; that is, it would be more than 1 per cent greater instead of being 1.25 per cent smaller, as when the wire is 0.08 cm in diameter. It is evident that the summation formula, which takes account of the actual size and position of the wires on the coil, gives the true values and that the current sheet formula, which is very exact for a current sheet or for a winding of thin strip which is equivalent to a current sheet, can not be applied without modification to a winding of round wires. Recognizing the fact that the two cases are not identical, Coffin applies a correction by reducing the length of the solenoid, supposing the equivalent current sheet to extend only to the centers of gravity of the outer semicircles of the end wires; that is, from a to b, Fig. 3. In this case the length of the winding would be shortened by 0.046 cm, making it 0.954 cm instead of 1.0 cm. Using this value of b in formula (1), we find the self-inductance of the coil to be

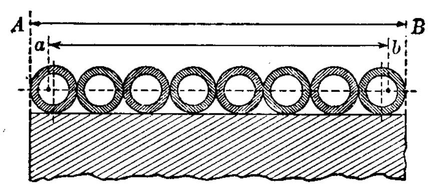

Fig. 3.

$$L = 4\pi a \times 100 \left\{\left(1 + \frac{(.954)^2}{20{,}000}\right) \log_e \frac{200}{.954} + \frac{(.954)^2}{80{,}000} - 0.5\right] \quad (6)$$
$$= 4\pi a \times 484.567$$

This is 4.707 (about 1 per cent) more than the former value, which is already too great. Thus, the correction makes the matter worse. It is evident that a series of round wires can not be regarded as equivalent to a current sheet, no matter how fine the wires are; for the magnetic field is very intense close to the wires, and the smaller the wire the stronger is the field and the greater is the self-inductance. For the case of mutual inductance between two coils at a distance from each other, the size of the wires is, of course, immaterial (if not too great), and hence single-layer coils, so far as mutual inductance is concerned, are equivalent to current sheets.

3. SELF-INDUCTANCE OF SINGLE TURNS OF THIN STRIP.

In order properly to compare the results given by the summation formula with the formula for the current sheet, we may apply the former to the case of a winding of very thin tape, to which the latter applies strictly. Let the tape be 1 mm wide, of infinitesimal thickness, and let 10 turns be applied to the cylinder of radius 25 cm. The length b is then 1 cm, and the value found above by formula (1) is accurate, namely,

$$\frac{L}{4\pi a}=479.8595.$$

To find 10 L_1, the first term of the summation formula, we use equation (5), replacing M by L and b by R, the geometric mean distance of the strip from itself; for the self-inductance of a conductor is equal to the mutual inductance of two filaments having a distance apart equal to the geometric mean distance of the conductor from itself. The g. m. d. R of a flat strip of negligible thickness is 0.223130 of its breadth.[5] Hence

$$L_1=4\pi a\left[\left(1+\frac{3}{16}\frac{R^2}{a^2}\right)\log\frac{8a}{R}-2-\frac{1}{16}\frac{R^2}{a^2}\right] \tag{7}$$

$$=4\pi a\left[\log\frac{8a}{R}-2\right],$$

neglecting terms in $\frac{R^2}{a^2}$, which here amount to less than one part in a million.

[5] Maxwell II, § 692.

Here $a=25$ cm and $R=.0223130$ cm. Substituting these numerical values, we find

$$\frac{10\,L_1}{4\pi a}=71.0090.$$

4. THE GEOMETRIC MEAN DISTANCE OF ONE STRIP FROM ANOTHER.

In order to calculate the mutual inductance of the several strips upon one another it is necessary to know their geometric mean distance from one another. Let Fig. 4 represent the section of the first and third turns of the winding, having a width a. Find first the g. m. d. of a point P distant b from the end taken as origin, from the line CD. Then

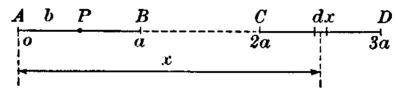

Fig. 4.

$$a \log R=\int_{2a}^{3a}\log(x-b)\,dx=\Big[(x-b)\log(x-b)-(x-b)\Big]_{2a}^{3a}$$

$$=(3a-b)\log(3a-b)-(2a-b)\log(2a-b)-a \tag{8}$$

where R is the g. m. d. of the point P from the line CD. To find the g. m. d. of all points in AB from the line CD we integrate again, this time along the line AB, first changing b in (8) to x. Thus,

$$a^2\log R_2=\int_0^a(3a-x)\log(3a-x)\,dx-\int_0^a(2a-x)\log(2a-x)\,dx$$

$$-\int_0^a a\,dx$$

$$=9\frac{a^2}{2}\log 3a-4a^2\log 2a+\frac{a^2}{2}\log a-3\frac{a^2}{2}$$

$$\text{or}\ \log R_2=\frac{9}{2}\log 3a-4\log 2a+\frac{1}{2}\log a-\frac{3}{2} \tag{9}$$

If we consider the first and fourth strips, the limits of integration would be $3a$ and $4a$, and o and a, respectively, and we should get

$$\log R_3=\frac{16}{2}\log 4a-9\log 3a+\frac{4}{2}\log 2a-\frac{3}{2} \tag{10}$$

The general expression is

$$\log R_n = \frac{(n+1)^2}{2}\log(n+1)a - n^2\log na + \frac{(n-1)^2}{2}\log(n-1)a - \frac{3}{2} \quad (11)$$

where na is the distance from the center of one element to the center of the other, a being the breadth of each element. Making n in the general expression equal to 0, 1, 2, 3, 4, etc., successively, we may find the values of the geometric mean distances of the first strip from the other nine.

The above formula is, however, not well adapted to numerical calculation when n is large, as the logarithm of R is the difference between large positive and negative terms, and unless the latter are calculated with extreme accuracy there is likely to be an appreciable error in the differences. The expression for log R may, however, be transformed into a series well adapted to numerical calculation for all values of n greater than unity.

Expanding the coefficients of equation (11) and recombining the terms we have, putting a equal to unity,

$$\log R_n = \frac{n^2}{2}\log(n^2-1) - \frac{n^2}{2}\log n^2 + n\log\frac{n+1}{n-1} + \frac{1}{2}\log(n^2-1) - \frac{3}{2}$$

$$= \frac{n^2}{2}\log\left(\frac{n^2-1}{n^2}\right) + \frac{1}{2}\log\left(\frac{n^2-1}{n^2}\right) + \frac{1}{2}\log n^2 + n\log\frac{n+1}{n-1} - \frac{3}{2}$$

$$= \frac{n^2+1}{2}\log\left(1-\frac{1}{n^2}\right) + n\log\left(\frac{n+1}{n}\right) - n\log\left(\frac{n-1}{n}\right) + \log n - \frac{3}{2} \quad (12)$$

Expanding the first three terms on the right of equation (12),

$$\log R_n = \log n - \frac{3}{2} - \frac{n^2+1}{2}\left(\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \frac{1}{4n^8} + \cdots\cdots\right)$$
$$+ 2n\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} + \cdots\cdots\right)$$

$$= \log n - \frac{3}{2} - \left(\frac{1}{2} + \frac{1}{4n^2} + \frac{1}{6n^4} + \frac{1}{8n^6} + \frac{1}{10n^8} + \cdots\cdots\right)$$
$$- \left(+\frac{1}{2n^2} + \frac{1}{4n^4} + \frac{1}{6n^6} + \frac{1}{8n^8} + \cdots\cdots\right)$$
$$+ \left(2 + \frac{2}{3n^2} + \frac{2}{5n^4} + \frac{2}{7n^6} + \frac{2}{9n^8} + \cdots\cdots\right)$$

$$\therefore \log R_n = \log n - \left(\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \cdots\cdots\right) \quad (13)$$

The coefficients of the denominators of the series are formed as follows:

$$\frac{1}{2}+\frac{1}{4}-\frac{2}{3}=\frac{1}{12}$$
$$\frac{1}{4}+\frac{1}{6}-\frac{2}{5}=\frac{1}{60}$$
$$\frac{1}{6}+\frac{1}{8}-\frac{2}{7}=\frac{1}{168} \text{ etc.}$$

The law of formation is evident, and any number of values can readily be calculated. The series is, however, *very convergent* for all values of n greater than 1. Thus when $n=2$, five terms are sufficient to give an accurate value of R_2. These terms are as follows:

$\log R_2 = \log 2 - (.0208333 + .0010417 + .0000930 + .0000108 + .0000015)$ whence $R_2 = 1.95653$.

When n equals 3 or 4, four terms suffice; when n is 5 or 6, three terms suffice, and for larger values of n two or even one term is sufficient. Thus—

$$\log R_5 = \log 5 - (.0033333 + .0000267 + .0000004)$$
$$\log R_8 = \log 8 - (.0013021 + .0000041)$$
$$\log R_{14} = \log 14 - (.0004252 + .0000004)$$
$$\log R_{24} = \log 24 - (.0001447 + .0000000)$$

The following values of the geometric mean distances (calling a unity) for the case under consideration were thus found:

$R_0 = 0.22313$	$R_5 = 4.98323$
$R_1 = 0.89252$	$R_6 = 5.98610$
$R_2 = 1.95653$	$R_7 = 6.98806$
$R_3 = 2.97171$	$R_8 = 7.98957$
$R_4 = 3.97890$	$R_9 = 8.99076$

5. CALCULATION OF THE MUTUAL INDUCTANCES FOR THE ASSUMED WINDING OF FLAT STRIP.

We can now calculate the last nine terms of formula (3), using formula (5), but putting the above values of R successively in place of the values of b, which in the case of round wires were 0.1, 0.2, 0.3 . . . . cm.

Thus, $$M_{12}=4\pi a\left\{\left(1+\frac{3}{16}\frac{(.089252)^2}{625}\right)\log\frac{200}{.089252}-2-\frac{1}{16}\frac{(.089252)^2}{625}\right\}$$

$$=4\pi a\times 5.71463 \qquad (14)$$

and $\frac{18M_{12}}{4\pi a}=102.8633.$

Proceeding in this way we find the values for the ten terms of equation (3). These values are given in column 2 of Table I. The values previously found for these terms for the winding of round wire are given in column 3. The differences between the corresponding terms are given in column 4, the sum of these differences being 6.0014, or 1.25 per cent, which represents the error of the current sheet formula when applied to this particular winding of round wire.

TABLE I.

	For Strip	For Round Wires	Differences
10 L_1	71.0090	67.6720	3.3370
18 M_{12}	102.8634	100.8166	2.0468
16 M_{13}	78.8771	78.5254	.3517
14 M_{14}	63.1671	63.0344	.1327
12 M_{15}	50.6421	50.5787	.0634
10 M_{16}	39.9525	39.9189	.0336
8 M_{17}	30.4965	30.4779	.0186
6 M_{18}	21.9449	21.9346	.0103
4 M_{19}	14.0950	14.0898	.0052
2 M_{110}	6.8120	6.8099	.0021
Total	479.8596	473.8582	6.0014

Thus, we have the value of $\frac{L}{4\pi a}$ for flat strips, by the two formulæ, as follows:

By the current sheet formula (1), 479.8595
By the summation formula (4), 479.8596

The difference between these results, amounting to less than one part in a million, is inappreciable. This discrepancy is of quite a different order of magnitude from the difference found between the current sheet formula and the summation formula for the case of round wires amounting to over 1.25 per cent. It is thus seen that this method of deducing the self-inductance of a coil by the method of summation is a practicable one, and in the case of a coil wound with flat strip it leads to correct results. There is no reason why it should not be equally exact in the case of round wires covered by insulation. We may therefore be sure that the value 473.8582 for the coil of ten turns of round wires is an accurate value for that case.

6. COIL OF LARGE NUMBER OF TURNS.

When the coil has a large number of turns of wire, it becomes impracticable to use the summation formula, because of the large number of terms to calculate. Thus, in the inductance standard of the Bureau of Standards having 661 turns there would be 661 terms to calculate. But we may calculate the differences shown in Table I, between the self and mutual inductances for round wires and for flat strip, and apply their sum as a correction to the value found by the current sheet formula, and so obtain the desired value with a moderate amount of labor; for the differences become very small after the distance becomes appreciable. Thus, at 1 cm in the above case the difference for a single pair of wires was only 1 per cent of its value at 1 mm. In calculating these differences in the case of the large coil, we should of course stop as soon as the difference becomes inappreciable. The standard of inductance of the Bureau of Standards has three sections, which may be used singly or in combinations. Thus, there are six different cases. The number of turns and length of each coil is given in Table II, together with the value of the inductances as calculated by the current sheet formula.

The mean radius of the coils is 27.0862 cm. The wire is round and has a diameter of 0.0634 cm bare, 0.0694 cm covered. As already indicated, the above values of the inductances, calculated by formulæ which are correct for a winding which is equivalent to a current sheet, are too great for a winding of round wires, in which the thickness of the insulation is small. We have found Coffin's correction to be wrong, as it makes the corrected value larger instead of smaller than the value for a current sheet. We shall now

TABLE II.

Coil	No. of Turns	Length	Inductance by Current-Sheet Formulæ
1	221	15.3347 cm	0.0361941 henry
2	251	17.3565 "	.0441703 "
3	189	13.1945 "	.0282220 "
1+2	472	32.6912 "	.112722 "
2+3	440	30.5510 "	.101810 "
1+2+3	661	45.8857 "	.179615 "

proceed to calculate the correction that must be applied to the above values to give the true values for this particular winding.

This correction consists of two parts and may be written as follows:

$$L_s - L = \Delta L_1 + \Delta M$$

where L_s is the value of the inductance calculated from the current sheet formula, L is the true value of the inductance, ΔL_1 is the correction depending on the self-inductance of the n single turns of the coil, and ΔM is the correction depending on the mutual inductances of each of the n turns on the $(n-1)$ other turns. The self-inductance of any coil of n turns may therefore evidently be written

$$L = L_s - \Delta L_1 - \Delta M$$

the corrections ΔL_1 and ΔM being subtracted from the value of the self-inductance given by the current sheet formula.

The total self-inductance of any coil of n turns may also be written

$$L = \Sigma L_1 + \Sigma M$$

where ΣL_1 is the sum of the self-inductances of all the n turns taken separately, and ΣM is the sum of the mutual inductances of each of the n turns on the $(n-1)$ other turns. The first term of the correction, ΔL_1, is thus the difference between the value of ΣL_1 for a winding of flat strip which would exactly represent the current sheet and its value for the actual winding of round insulated wire; while the second term is the difference between ΣM for the winding of strip and the actual winding of wire.

7. TO CALCULATE THE CORRECTION ΔL.

Maxwell's formula for the self-inductance of a single circular turn of round wire, which is practically equivalent to Wien's, is derived from the formula for the mutual inductance of two parallel coaxial circles, by replacing b the distance apart of their planes by R the geometric mean distance of the section, which in this case is a circle. Thus

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{R^2}{a^2}\right)\log\frac{8a}{R}-2-\frac{1}{16}\frac{R^2}{a^2}\right\} \tag{17}$$

The geometric mean distance R for the circular section of straight wire is $\rho\epsilon^{-1/4}=.778801\rho$ where ρ is the radius of the section. This is not quite exact where the conductor is a circle, but is sufficiently exact when a is large and ρ relatively small. Wien's formula is derived directly by integrating the expression for the mutual inductance of two parallel circles (of infinitesimal section) twice over the area of the circular cross section of the conductor. By comparing the results of the two formulæ we may get an idea of the magnitude of the error arising from using the g. m. d. of the circle as the same as that for a rectilinear conductor. If $a=25$ cm and $\rho=.05$ cm, L, the self-inductance of one turn of wire, is 654.40537π cm by Wien's formula and 654.40533π cm by Maxwell's. The difference is inappreciable. We need not hesitate, therefore, to use Maxwell's formula in the present case, where ρ is less than 0.04 cm.

The self-inductance L_1 of a round wire of radius ρ is given by formula (17), where R will be written R_1 and will have a value 0.778801ρ. Similarly, the self-inductance L_2 of a single turn of flat strip of width d wound on a circle of radius a will be given by the same equation, except that R (here written R_2) will have a value $0.223130d$. The difference between the two will be

$$L_2-L_1=4\pi a\log\frac{R_1}{R_2}, \tag{18}$$

neglecting small quantities, which here amount to less than one part in a million. Where the bare wire has a diameter 0.0634, $\rho=.0317$. The strip has a width D equal to the diameter of the covered wire; $=0.0694$.

Hence

$$R_1 = .778801\ \rho$$
$$\text{and } R_2 = .223130\ D$$
$$\therefore \frac{R_1}{R_2} = 1.594.$$

Thus the excess of the self-inductance of the n turns taken separately of a coil wound with flat strip of width 0.0694 cm over the self-inductance of the same number of turns of round wire of 0.0634 cm diameter (0.0694 covered) is

$$\varDelta L_1 = 4\pi an \log_e \frac{R_1}{R_2}$$
$$\text{or, } \varDelta L_1 = 4\pi an \log_e 1.594. \qquad (19)$$

In the standard of inductance under consideration $a = 27.0862$ cm, and the number of turns of wire in each of the six sections is given in Table II. Substituting these values in equation (19), we have the following values of the corrections $\varDelta L$:

Coil 1	$\varDelta L_1 =$	35073 cm
" 2	$\varDelta L_2 =$	39833 "
" 3	$\varDelta L_3 =$	29993 "
" 1+2	$\varDelta L_4 =$	74904 "
" 2+3	$\varDelta L_5 =$	69827 "
" 1+2+3	$\varDelta L_6 =$	104900 "

8. TO CALCULATE THE CORRECTION $\varDelta$M.

The corrections $\varDelta M$ are found in a similar manner. The mutual inductance of two parallel circles of round wire is given by equation (5), where b is the distance apart of the centers of the wires, which is also the geometric mean distance, supposing the radius a is large and b relatively small. In the case of the two strips, forming part of a current sheet, we have found the g. m. d. to be less than the distance apart of their centers. The mutual inductance of two such strips will therefore always be greater than that of two round wires, the distance apart of their centers being supposed the same in each case. The mutual inductance in the case of the wires will be,

$$M_1 = 4\pi a \left\{ \left(1 + \frac{3}{16} \cdot \frac{b^2}{a^2}\right) \log \frac{8a}{b} - 2 - \frac{1}{16} \cdot \frac{b^2}{a^2} \right\} \qquad (20)$$

and for the strip, putting kb for the geometric mean distance of the strips, where k is always less than unity,

$$M_2 = 4\pi a\left\{\left(1+\frac{3}{16}\cdot\frac{k^2b^2}{a^2}\right)\log\frac{8a}{kb} - 2 - \frac{1}{16}\cdot\frac{k^2b^2}{a^2}\right\} \tag{21}$$

The difference is

$$M_2 - M_1 = 4\pi a\left\{\log\frac{1}{k} + \frac{1}{16}\cdot\frac{b^2}{a^2}\left(1 - k^2 + 3k^2 \log\frac{8a}{kb} - 3\log\frac{8a}{b}\right)\right\} \tag{22}$$

In the above expression for $M_2 - M_1$, k does not differ from unity appreciably except where b is very small; that is, where the strips are very near together. Thus the second part of the expression is negligible in all cases, for when the coefficient $\frac{1}{16}\frac{b^2}{a^2}$ is more than 0.000001 (its value for $b = 1$ mm) k is so nearly unity that the quantity within the parentheses is very small. Thus, for $b = 1$ cm, the coefficient is 0.0001, $k = 0.999575$ and the quantity in the parentheses is about 0.001, so that the term amounts to 0.0000001 and can be neglected. The correction δM for any pair of wires is thus

$$\delta M = 4\pi a \log\frac{1}{k}. \tag{23}$$

Since the value of k depends upon the distance apart of the two turns of wire or strip under consideration, it is evident that there will be as many different terms as there are different distances. Thus

$$\delta M_1 = 4\pi a \log\frac{1}{k_1} = 4\pi a\delta_1$$

$$\delta M_2 = 4\pi a \log\frac{1}{k_2} = 4\pi a\delta_2, \text{ etc.} \tag{24}$$

where δM_1 is the correction for a pair of adjacent wires, the distance of their centers being D, the diameter of the covered wire, and Dk_1 their g. m. d.; δM_2 is the correction for a pair of wires distant $2D$, $2Dk_2$ being their g. m. d., etc. If there are n turns of wire on the coil we shall have

$$\Delta M = 2(n-1)\delta M_1 + 2(n-2)\delta M_2 + 2(n-3)\delta M_3 + \dots$$

$$= 8\pi a\ [(n-1)\delta_1 + (n-2)\delta_2 + (n-3)\delta_3 + \dots] \qquad (25)$$

In a coil of n turns there would be $(n-1)$ terms in this equation, but inasmuch as k rapidly approaches unity when the distance is increased, the correction terms decrease rapidly in value, so that only a limited number of terms need be calculated.

In Table III the consecutive values of the geometric mean distances are given up to R_9 and then every fifth value is given up to

TABLE III.

Geometric Mean Distances and Corrections Depending upon Them.

Geometric Mean Distances R	$k=\frac{R_n}{n}$	$\frac{1}{k}$	$\delta=\log_e\frac{1}{k}$
R_1 0.89252	0.89252	1.12042	0.11371
R_2 1.95653	.978265	1.02223	.02198
R_3 2.97171	.99057	1.00952	.00948
R_4 3.97890	.99473	1.00529	.00528
R_5 4.98323	.99665	1.00336	.00336
R_6 5.98610	.99768	1.00233	.00233
R_7 6.98806	.99829	1.00171	.00171
R_8 7.98957	.99869	1.00131	.00131
R_9 8.99076	.99897	1.00103	.00103
R_{14} 13.99405	.999575	1.000425	.000425
R_{19} 18.99531	.999753	1.000247	.000247
R_{24} 23.99653	.999855	1.000145	.000145
R_{29} 28.99724	.999905	1.000095	.000095
R_{34} 33.99754	.999928	1.000072	.000072
R_{39} 38.99785	.999945	1.000055	.000055
R_{44} 43.99811	.999957	1.000043	.000043
R_{49} 48.99828	.999965	1.000035	.000035
R_{99} 98.99920	.999992	1.000008	.000008

R_{49}, corresponding to the first and fiftieth turns of wire, respectively.

The second column gives the coefficients k obtained by dividing each geometric mean distance by the corresponding distance between centers of the wire, indicated by its subscript; thus, R_2 is divided by 2, etc. The third column gives the values of $\frac{1}{k}$ and the fourth column gives the corrections δ, the natural logarithms of $\frac{1}{k}$. These

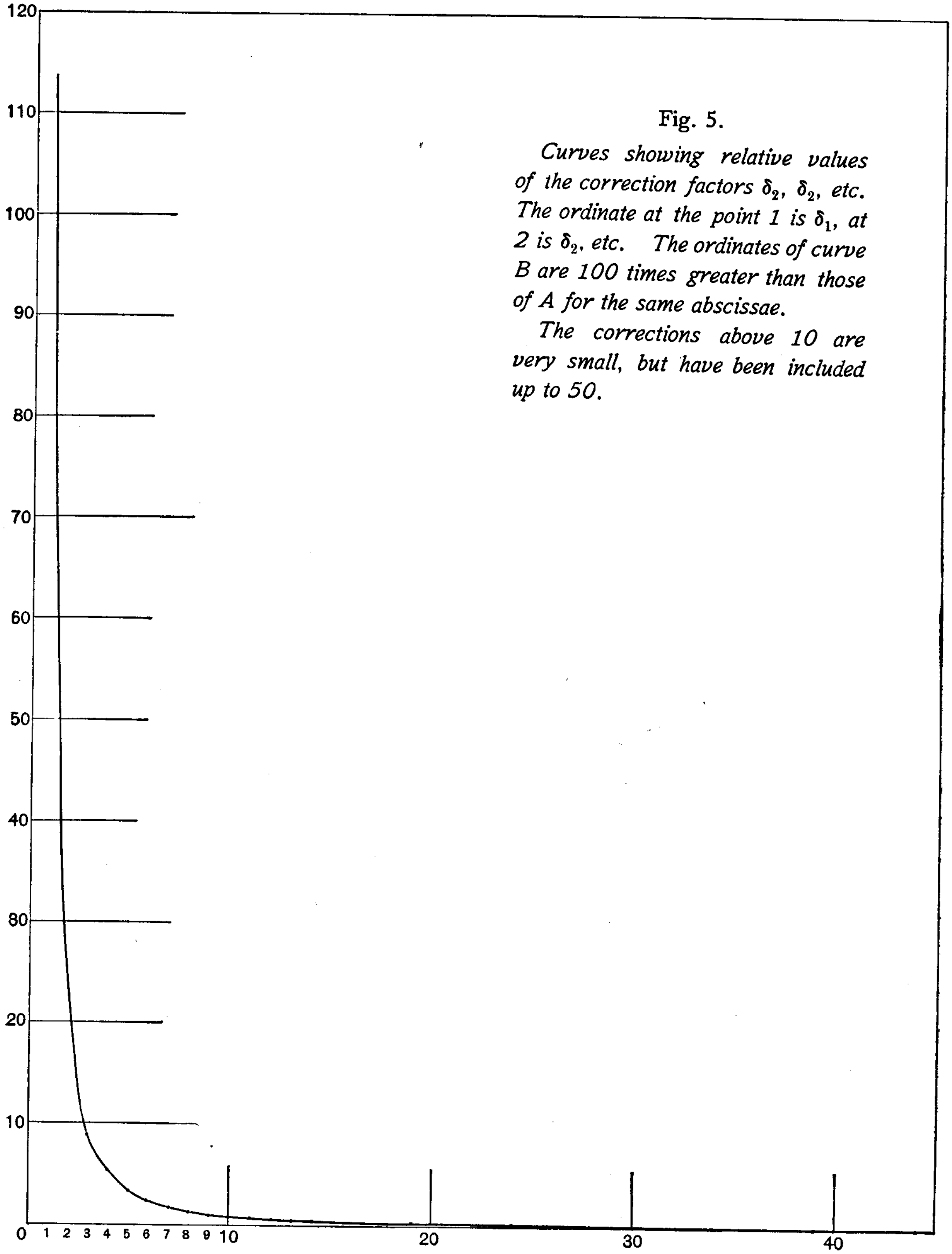

Fig. 5.

Curves showing relative values of the correction factors δ_2, δ_2, etc. The ordinate at the point 1 is δ_1, at 2 is δ_2, etc. The ordinates of curve B are 100 times greater than those of A for the same abscissae.

The corrections above 10 are very small, but have been included up to 50.

values are plotted in Fig. 5. The curve shows the values of δ up to $n = 45$. This curve shows how rapidly the quantity δ falls off as the distance increases. For the first and fiftieth wires it is only three ten-thousandths of its value for the first pair of wires, whereas the correction for the first and one-hundredth wire is less than one ten-thousandth of the first pair.

In Tables IV and V the products $(n-1)\delta_1$, $(n-2)\delta_2$, etc., are given

TABLE IV.

Corrections to Mutual Inductance.

B. S. Standard Coil. Sections 1, 2, 3.

Log $\frac{1}{k}$	Coil 1, n=221		Coil 2, n=251		Coil 3, n=189	
	n-1	Product	n-1	Product	n-1	Product
δ_1=.11371	220	25.016	250	28.428	188	21.377
δ_2=.02198	219	4.814	249	5.473	187	4.110
δ_3=.00948	218	2.066	248	2.351	186	1.763
δ_4=.00528	217	1.146	247	1.304	185	0.977
δ_5=.00336	216	0.726	246	0.827	184	0.618
δ_6=.00233	215	0.501	245	0.571	183	0.426
δ_7=.00171	214	0.366	244	0.417	182	0.311
δ_8=.00131	213	0.279	243	0.318	181	0.237
δ_9=.00103	212	0.218	242	0.249	180	0.185
$\Sigma\delta_{10-14}$=.00321	209	0.671	239	0.767	177	0.568
$\Sigma\delta_{15-19}$=.00163	204	0.332	234	0.381	172	0.280
$\Sigma\delta_{20-24}$=.00096	199	0.191	229	0.220	167	0.143
$\Sigma\delta_{25-100}$	....	0.493	....	0.576	......	0.405
	$\frac{\Delta M}{8\pi a}$	= 36.819		41.882		31.400
$8\pi a$=680.75, ∴ ΔM		= 25,064 cm		28,511 cm		21,375 cm

for the six sections of the N. B. S. standard of inductance, as indicated in equation (25). The sum of the terms and $8\pi a$ times the sum are given for each coil, the latter being the correction ΔM for mutual inductance. Each term is given separately up to δ_9, after that the sum of five consecutive terms is given, as $\Sigma\delta_{10-14}$, and finally the correction for all the turns from 25 to 100 is given in one. It will be noticed that the first four terms amount to about 90 per cent of the whole correction, and that the sum of the terms after δ_{20} amounts to only a little more than 1 per cent of the whole correction, or 1 part in 100,000 of the whole inductance.

The geometric mean distances given are those between parallel straight wires, and not parallel circles. But the quantity *k* is the ratio between the g. m. ds. of flat strips and round wires, and this

TABLE V.

Corrections to Mutual Inductance.

B. S. Standard Coil. Sections 1+2, 2+3, 1+2+3.

$\text{Log}\,\frac{1}{k}$	Coils 1+2, n=472		Coils 2+3, n=440		Coils 1+2+3, n=661	
	n−1 (n−2) etc.	Product	n−1 (n−2) etc.	Product	n−1 (n−2) etc.	Product
δ_1=.11371	471	53.557	439	49.919	660	75.049
δ_2=.02198	470	10.331	438	9.627	659	14.485
δ_3=.00948	469	4.446	437	4.143	658	6.238
δ_4=.00528	468	2.471	436	2.301	657	3.469
δ_5=.00336	467	1.569	435	1.462	656	2.204
δ_6=.00233	466	1.086	434	1.011	655	1.526
δ_7=.00171	465	0.795	433	0.740	654	1.118
δ_8=.00131	464	0.608	432	0.566	653	0.855
δ_9=.00103	463	0.477	431	0.444	652	0.672
$\Sigma\delta_{10-14}$=.00321	460	1.477	428	1.374	649	2.083
$\Sigma\delta_{15-19}$=.00163	455	0.742	423	0.689	644	1.050
$\Sigma\delta_{20-24}$=.00096	450	0.432	418	0.401	639	0.614
$\Sigma\delta_{25-100}$	....	1.180	....	1.095	....	1.700
	$\frac{\Delta M}{8\pi a}$ =	79.171		73.770		111.063
$8\pi a$=680.75, ΔM =		53,896 cm		50,219 cm		75,606 cm

can not be appreciably different for the case of parallel circles at moderate distances from its value for straight conductors. For the difference in the g. m. ds. is very small, and the difference in their ratios would be a small quantity of the second order. Hence the corrections calculated by (25) and given in Tables IV and V must be very exact.

In Table VI is given a summary of the values of the two corrections ΔL_1 and ΔM for the six sections of the standard coil, together with the values of the inductances given by the current sheet formula and the final corrected values. It will be noticed that the whole correction varies in amount from 1 part in 600 for the smaller section to 1 part in 1,000 for the whole coil. This is a very large quantity in a standard when we remember the extreme

TABLE VI.

Summary of Corrections ΔL_1 and ΔM for the Six Sections of the N. B. S. Standard of Inductance, and the Corrected Values of the Inductances

	Coil No. 1 n=221	Coil No. 2 n=251	Coil No. 3 n=189
Correction for self-inductance ΔL_1	35073 cm	39833 cm	29993 cm
Correction for mutual inductance ΔM	25064 "	28511 "	21375 "
Total correction ΔL	60137 "	68344 "	51368 "
Total correction ΔL	0.0000601 henry	0.0000683 henry	0.0000514 henry
Inductance by current sheet formula L_s	0.0361941 "	0.0441703 "	0.0282220 "
Corrected inductance L	0.0361340 "	0.0441020 "	0.0281706 "

	Coils 1+2 n=472	Coils 2+3 n=440	Coils 1+2+3 n=661
Correction for self-inductance ΔL_1	74904 cm	69827 cm	104900 cm
Correction for mutual inductance ΔM	53896 "	50219 "	75606 "
Total correction ΔL	128800 "	120046 "	180506 "
Total correction ΔL	0.000129 henry	0.000120 henry	0.000180 henry
Inductance by current sheet formula L_s	0.112722 "	0.101810 "	0.179615 "
Corrected inductance L	0.112593 "	0.101690 "	0.179435 "

precision with which the measurements of the dimensions were made and the high sensibility obtainable in measuring self-inductance.

9. THE CORRECTION TABLES.

It is possible to put these two correction terms into such form that they may be quickly applied to any single layer coil, by the aid of tables of constants. Since $0.77880\rho = 0.3894d$, where d is the diameter of the bare wire, equation (19) may be written

$$\Delta L_1 = 4\pi a n \log_e \frac{0.3894d}{0.22313D}$$

$$= 4\pi a n \log_e \left(1.7452\frac{d}{D}\right) \tag{26}$$

From equation (25) we have

$$\Delta M = 8\pi a \sum_{n-1}^{n-50} \left(n \log \frac{1}{k}\right)$$

The sum of these two terms may be written

$$\Delta L = \Delta L_1 + \Delta M = 4\pi a n [A+B] \tag{27}$$

where A stands for $\log_e \left(1.7452\frac{d}{D}\right)$ and B stands for $\frac{2}{n} \Sigma \left(n \log \frac{1}{k}\right)$, the summation being carried from $n=(n-1)$ to $n=1$ for coils of less than 50 turns and up to $(n-50)$ for coils of more than 50 turns.

The values of the constants A are given in Table VII with the ratios $\frac{d}{D}$ as arguments. D is the width of the current sheet corresponding to one turn of wire. If the wire is wound so that the consecutive turns are in contact, D is also the diameter of the insulated wire. The mean length of the coil divided by the number of turns gives the value of D to be used. The mean diameter of the bare wire is d. The two corrections ΔL_1 and ΔM are to be subtracted from L_s. When the ratio $\frac{d}{D}$ is less than about 0.57, A is negative and hence ΔL_1 is negative, and it is added; ΔM is always positive.

The values of the constant B are given in Table VIII with n the total number of turns in the coil as argument. By means of these two tables and equation (27) it is easy to find the correction for any particular case, as the following illustrations will show.

TABLE VII.

Values of Correction Term A, Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil.

$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1
1.00	0.5568	100	.70	0.2001	144
.99	.5468	101	.69	.1857	146
.98	.5367	103	.68	.1711	148
.97	.5264	104	.67	.1563	150
.96	.5160	105	.66	.1413	152
.95	.5055	106	.65	.1261	155
.94	.4949	107	.64	.1106	157
.93	.4842	108	.63	.0949	160
.92	.4734	109	.62	.0789	163
.91	.4625	110	.61	.0626	166
.90	.4515	112	.60	.0460	168
.89	.4403	113	.59	.0292	171
.88	.4290	114	.58	.0121	174
.87	.4176	116	.57	— .0053	177
.86	.4060	117	.56	— .0230	180
.85	.3943	118	.55	— .0410	184
.84	.3825	120	.54	— .0594	187
.83	.3705	121	.53	— .0781	190
.82	.3584	123	.52	— .0971	194
.81	.3461	124	.51	— .1165	198
.80	.3337	126	.50	— .1363	
.79	.3211	127	.50	— .1363	1053
.78	.3084	129	.45	— .2416	1178
.77	.2955	131	.40	— .3594	1335
.76	.2824	133	.35	— .4928	1542
.75	.2691	134	.30	— .6471	1823
.74	.2557	136	.25	— .8294	2232
.73	.2421	138	.20	—1.0526	2877
.72	.2283	140	.15	—1.3403	4054
.71	.2143	142	.10	—1.7457	
.70	.2001				

TABLE VIII.

Values of the Correction Term B, Depending on the Number of Turns of Wire on the Single-Layer Coil.

Number of Turns	B	Number of Turns	B
1	0.0000	50	0.3186
2	.1137	60	.3216
3	.1663	70	.3239
4	.1973	80	.3257
5	.2180	90	.3270
6	.2329	100	.3280
7	.2443	125	.3298
8	.2532	150	.3311
9	.2604	175	.3321
10	.2664	200	.3328
15	.2857	300	.3343
20	.2974	400	.3351
25	.3042	500	.3356
30	.3083	600	.3359
35	.3119	700	.3361
40	.3148	800	.3363
45	.3169	900	.3364
50	.3186	1000	.3365

10. EXAMPLES ILLUSTRATING THE USE OF THE CORRECTION TABLES.

Example 1.—Coil of 10 turns, radius 25 cm, length 1 cm, diameter of insulated wire 0.1 cm $=D$, diameter of bare wire 0.08 cm $=d$; thus $\frac{d}{D}=0.8$.

$$\begin{aligned} \text{From Table VII, } A &= 0.3337 \\ \text{" \quad " \quad VIII, } B &= 0.2664 \\ \hline A+B &= 0.6001 \\ n(A+B) &= 6.001 \\ \therefore \Delta L &= 4\pi a \times 6.001. \end{aligned}$$

This value of ΔL is to be subtracted from L_s to obtain the true value L of the self-inductance.

This is the value found already (p. —), where the correction was determined by calculating L by the summation formula (2) and L_s by the current sheet formula (1).

Example 2.—Coil of 50 turns, radius $a=20$ cm, length $b=5$ cm, $D=0.1$ cm, $d=0.075$ cm; thus $\frac{d}{D}=0.75$.

By formula (1)

$$L_s=4\pi a\times 2500\left\{\left(1+\frac{25}{32\times 400}\right)\log_e \frac{160}{5}+\frac{25}{128\times 400}-.50\right\}$$

$$=4\pi a\times 2500\left\{\left(1+\frac{1}{512}\right)\log_e 32+\frac{1}{2048}-.50\right\}$$

$$L_s=4\pi a\times 2500\times 2.972945=4\pi a\times 7432.36.$$

From Table VII, $A=0.2691$
" " VIII, $B=0.3186$

$$A+B=0.5877$$
$$n(A+B)=29.39$$

$$\therefore\ L=L_s-\Delta L=4\pi a\ (7432.36-29.39)$$
$$4\pi a=251.3274$$
$$\therefore\ L=1860570 \text{ cm}$$
$$=1.86057 \text{ millihenrys.}$$

The correction to L_s here amounts to 0.4 per cent.

Example 3.—As an extreme case to test the method we may calculate the self-inductance of a single turn of wire. Let us take the particular case already calculated by Wien's and Maxwell's formulæ, (4) and (7), page —. The radius $a=25$ cm, the diameter of the bare wire $=1$ mm. We may now assume that the wire is covered and that the diameter D is 2 mm. Then $\frac{d}{D}=0.5$. In using Rayleigh's current sheet formula we take the length of the equivalent current sheet as equal to D. We thus have

$$L_s=4\pi a\left\{\left(1+\frac{.04}{32\times 625}\right)\log_e \frac{200}{0.2}+\frac{.04}{128\times 625}-0.5\right\}$$

$$=4\pi a\left\{\left(1+\frac{1}{500,000}\right)6.907755+\frac{1}{2,000,000}-0.5\right\}$$

$$=4\pi a\times 6.40777.$$

From Tables VII and VIII $A=-0.1363$ and $B=0$. Thus, since $n=1$, $\Delta L=4\pi a\times(-0.1363)$, and being negative is added to L_s. Hence

$$L=4\pi a\,(6.40777+0.1363)=4\pi a\times 6.54407$$
$$=654.407\pi.$$

This is practically identical with the values given by the other formulæ (p. —), the slight difference being due to the fact that the correction term A is carried only to four places of decimals.

If we had taken the bare wire of diameter 0.1 cm as equivalent to a current sheet 0.1 cm long in the above formulæ for L_s, we should have obtained a different value for L_s, but in that case $\frac{d}{D}$ would be unity and A would be $+.5568$. The resulting value of L would, however, be the same as before.

Example 4.—Take the first section of the B. S. standard. Here $n=221$, $d=.0634$, $D=.0694$. Hence $\frac{d}{D}=.9135$.

$$\begin{aligned} \text{From Table VII } A &= 0.4663 \\ \text{“ \quad “ VIII } B &= 0.3332 \\ A+B &= 0.7995 \\ n\,(A+B) &= 176.690 \end{aligned}$$

$$4\pi a=340.375$$
$$\therefore\ \Delta L=4\pi a n\,(A+B)=60{,}141 \text{ cm}$$
$$=.0000601 \text{ henry.}$$

This is practically identical with the correction calculated directly for this section (Table VI).

Taking the whole coil, for which $n=661$, we have

$$\begin{aligned} A &= 0.4663 \\ B &= 0.3360 \\ A+B &= 0.8023 \\ n\,(A+B) &= 530.32 \\ 4\pi a &= 340.375 \\ \Delta L &= 4\pi a n\,(A+B)=180{,}508 \text{ cm} \\ &= .0001805 \text{ henry} \end{aligned}$$

which is the same value found previously for ΔL.

These examples are sufficient to illustrate the use and the accuracy of the correction Tables VII and VIII. By their use an accurate value of the self-inductance of a single layer coil of any number of turns can be calculated, if the proper current sheet formula is employed. Rayleigh's formula (1) already used is the most convenient one for short coils; that is, for coils whose length is small compared with the radius. Coffin's formula is an extension of Rayleigh's, and may be used where the length is too great to omit terms in $\frac{b^4}{a^4}$, $\frac{b^6}{a^6}$, and $\frac{b^8}{a^8}$ (b being the length and a the radius). Lorenz's formula is an absolute one, and may be used for coils of any length, being more exact than Coffin's for coils whose length is as great as the diameter, but agreeing with Coffin's very exactly for all lengths up to $b=a$.

For convenience of reference I here give these three formulæ.

1. Rayleigh's formula:

$$L_s=4\pi a n^2\left\{\log\frac{8a}{b}-\frac{1}{2}+\frac{b^2}{32a^2}\left(\log\frac{8a}{b}+\frac{1}{4}\right)\right\} \tag{28}$$

2. Coffin's formula:

$$L_s=4\pi a n^2\left\{\log\frac{8a}{b}-\frac{1}{2}+\frac{b^2}{32a^2}\left(\log\frac{8a}{b}+\frac{1}{4}\right)-\frac{1}{1024}\frac{b^4}{a^4}\left(\log\frac{8a}{b}-\frac{2}{3}\right)\right.$$
$$\left.+\frac{10}{131072}\frac{b^6}{a^6}\left(\log\frac{8a}{b}-\frac{109}{120}\right)-\frac{35}{4{,}194{,}304}\frac{b^8}{a^8}\left(\log\frac{8a}{b}-\frac{431}{420}\right)\right\} \tag{29}$$

3. Lorenz's formula:

$$L_s=\frac{4\pi}{3}\frac{n^2}{b^2}\left\{d\,(4a^2-b^2)\,E+db^2\,F-8a^3\right\} \tag{30}$$

In the above formulæ a = radius of the coil and b = length of the coil, the length being the *mean over-all length including the insulation on the first and last wires.* In formula (3), d is the diagonal of the coil $=\sqrt{4a^2+b^2}$ and E and F are the complete elliptical integrals of the first and second kinds, respectively, to modulus k, where $k=\frac{2a}{d}=\frac{2a}{\sqrt{4a^2+b^2}}$. The subscript s is attached to L in each case as a reminder that each is a current sheet formula, and the value of the inductance must in every case be corrected by formula (27) and Tables VII and VIII in order to give the true inductance of a winding of round wires. A winding of square or rectangular wire would also require correction, the only winding for which the formulæ are correct being a winding of strip of infinitesimal thickness in which the edges of the strip come together without making electrical contact, and so fulfilling the current sheet conditions assumed in deriving the formula.

In a subsequent paper I shall discuss the case of coils having more than one layer.