2 Hertz Potentials

2.1 Introduction

A general feature of classical electrodynamics is the fact that an electromagnetic field must be a solution of Maxwell's equations. Therefore many theoretical considerations on the structure of Maxwell's equations exist. The analysis of an electromagnetic field is often facilitated by the use of auxiliary functions known as potential functions (scalar, vectorial, tensorial) [1, p.23]. These are solutions of partial differential equations. The partial differential equations are such, that solving them for the potential functions is equivalent to the more tedious task of solving Maxwell's equations directly [2].

The most elegant approach to this is the field representation in terms of Green's tensors. However, this treatment has the fundamental disadvantage that tensor differential equations have to be solved. It is only for special kinds of media that Green's tensors can be reduced to scalar Green's functions [2].

It was shown by Hertz that an arbitrary electromagnetic field in a (source free) homogeneous linear isotropic medium can be defined in terms of a single vector potential $\vec{\Pi}$ [1, p.28]. *The Hertz vector potential* notation is an efficient mathematical formalism for solving electromagnetic problems. As will be shown, Hertz vector potential can be reduced to a set of two scalar potentials, which are solutions of Helmholtz's equations, for any orthogonal curvilinear coordinate system. These solutions are independent only in the case of an isotropic medium [2]. Note that at present the Hertz potential notation has been extended in order to take into account sources contained in the medium [1, pp. 30-32 & pp. 430-431]. However, this can not be done in a straightforward manner. The current and charge densities first need to be expressed in terms of *an electric polarization vector* $\vec{P}$ using the formulas: $\vec{J} = \frac{\partial \vec{P}}{\partial t}$ and $\rho = -\vec{\nabla} \cdot \vec{P}$.

A lot of present day textbooks on the subject of electromagnetism rely heavily on *the magnetic vector potential* $\vec{A}$ and *the scalar potential* ϕ, also often called the mixed potential method. The main advantage of this method is the fact that the two Helmholtz's equations that result from it (one vectorial and one scalar), directly take into account any current or charge sources lying in the medium. This is in contrast with the Hertz vector potential method where, as has been explained in the previous paragraph, the scalar potentials are more closely connected to the field intensities $\vec{E}$ and $\vec{H}$.

The magnetic vector potential $\vec{A}$ and the scalar potential ϕ are related to the Hertz vector potential as follows [1, pp. 28-29]: $\vec{A} = \mu\varepsilon \frac{\partial \vec{\Pi}}{\partial t}$ and $\phi = -\vec{\nabla} \cdot \vec{\Pi}$, provided that $\vec{A}$ is defined by $\vec{B} = \vec{\nabla} \times \vec{A}$ and not $\vec{H} = \vec{\nabla} \times \vec{A}$. (The latter definition is more common in East European countries.)

The big strength of the Hertz vector potential method lies with the fact that there is no need to check whether the solutions of the two scalar partial differential equations are solutions of the posed problem. This is clearly not the case with the mixed potential method [3, p. 679]. For problems situated in source free media, this property of the Hertz vector potential method far outweighs the advantages of any other method. This also explains why, in this text, the Hertz vector potential method is preferred over the mixed potential method.

In recent years, a lot of research effort went into the development of potential formulations for anisotropic, gyrotropic, chiral and spatially inhomogeneous media [2]. For a detailed discussion on scalar Hertz potentials for bigyrotropic media see [4].

2.2 Hertz's Wave Equation for Source Free Homogeneous Linear Isotropic Media

Assuming $e^{j\omega t}$ time dependence, *Hertz's wave equation* for a source free homogeneous linear isotropic medium, independent of the coordinate system, is [3, p. 729]

$$\nabla^2 \vec{\Pi} + k^2 \vec{\Pi} = 0 \tag{1}$$

where $\nabla^2 \vec{v} \equiv \vec{\nabla}\left(\vec{\nabla} \cdot \vec{v}\right) - \vec{\nabla} \times \vec{\nabla} \times \vec{v}$ ($\vec{v}$ is any vector) [1, p. 25], [3, p. 95]

and $k^2 = -j\omega\mu(\sigma + j\omega\varepsilon) = \varepsilon\mu\omega^2 - j\omega\mu\sigma$.

(k is *the complex wave number* of the surrounding medium.)

Hertz's wave equation for source free homogeneous linear isotropic media (1) has two types of independent solutions: $\vec{\Pi}_e$ and $\vec{\Pi}_m$.

These result in independent sets of *E-type waves*

$$\vec{H} = (\sigma + j\omega\varepsilon)\vec{\nabla} \times \vec{\Pi}_e, \tag{2a}$$

$$\vec{E} = k^2 \vec{\Pi}_e + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\Pi}_e\right), \tag{2b}$$

and *H-type waves*, respectively [3, p. 729]

$$\vec{E} = -j\omega\mu\, \vec{\nabla} \times \vec{\Pi}_m, \tag{3a}$$

$$\vec{H} = k^2 \vec{\Pi}_m + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\Pi}_m\right). \tag{3b}$$

Note that throughout this text, permittivity ε will be treated as a complex quantity with two distinct loss contributions [5]

$$\varepsilon = \varepsilon' - j\varepsilon'' - j\frac{\sigma}{\omega}$$

where $-j\varepsilon''$ is the loss contribution due to molecular relaxation

and $-j\frac{\sigma}{\omega}$ is the conduction loss contribution. (The conductivity σ is measured at DC.)

However, in practice it is not always possible to make this distinction. This is often the case with metals and good dielectrics. In those cases all losses can be treated as though being entirely due to conduction or molecular relaxation, respectively.

Above relations follow from

$$\vec{\nabla} \times \vec{H} = j\omega\vec{D} + \vec{J} = j\omega(\varepsilon' - j\varepsilon'')\vec{E} + \sigma\vec{E}.$$

The loss tangent of a dielectric medium is defined by

$$\tan\delta \equiv \frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'}.$$

Permeability μ has only one loss contribution due to hysteresis: $\mu = \mu' - j\mu''$.

2.3 Hertz's Wave Equation in Orthogonal Curvilinear Coordinate Systems with Two Arbitrary Scale Factors

Consider a right-hand orthogonal curvilinear coordinate system with *curvilinear coordinates* (u_1, u_2, u_3). *Scale factor* h_1 equals one and scale factors h_2 and h_3 can be chosen arbitrary.

(A detailed explanation of what curvilinear coordinates and scale factors are, can be found in [1, pp. 38-59] and [6, pp. 124-130], together with definitions of gradient, divergence, curl and Laplacian for such coordinate systems.)

Hertz's vector wave equation for source free homogeneous linear isotropic media (1) can be reduced to a scalar wave equation [3, pp. 729-730] by making use of the definitions given in [1, pp. 49-50]

$$\frac{\partial^2 \Pi}{\partial u_1^2} + \frac{1}{h_2 h_3}\frac{\partial}{\partial u_2}\left(\frac{h_3}{h_2}\frac{\partial \Pi}{\partial u_2}\right) + \frac{1}{h_2 h_3}\frac{\partial}{\partial u_3}\left(\frac{h_2}{h_3}\frac{\partial \Pi}{\partial u_3}\right) + k^2\,\Pi = 0 \tag{4}$$

with $\vec{\Pi} = \Pi(u_1, u_2, u_3)\vec{e}_1$. (5)

$\vec{e}_1$ is in the unit vector in the u_1-direction.

The field components of the E-type waves are obtained by introducing (5) into (2a+b)

$$\begin{aligned}
E_1 &= k^2\,\Pi_e + \frac{\partial^2 \Pi_e}{\partial u_1^2}; \quad H_1 = 0, \\
E_2 &= \frac{1}{h_2}\frac{\partial^2 \Pi_e}{\partial u_1 \partial u_2}; \quad H_2 = \frac{(\sigma + j\omega\varepsilon)}{h_3}\frac{\partial \Pi_e}{\partial u_3}, \\
E_3 &= \frac{1}{h_3}\frac{\partial^2 \Pi_e}{\partial u_1 \partial u_3}; \quad H_3 = -\frac{(\sigma + j\omega\varepsilon)}{h_2}\frac{\partial \Pi_e}{\partial u_2}.
\end{aligned} \tag{6}$$

The field components of the H-type waves are obtained by introducing (5) into (3a+b)

$$\begin{aligned}
H_1 &= k^2\,\Pi_m + \frac{\partial^2 \Pi_m}{\partial u_1^2}; \quad E_1 = 0, \\
H_2 &= \frac{1}{h_2}\frac{\partial^2 \Pi_m}{\partial u_1 \partial u_2}; \quad E_2 = -\frac{j\omega\mu}{h_3}\frac{\partial \Pi_m}{\partial u_3}, \\
H_3 &= \frac{1}{h_3}\frac{\partial^2 \Pi_m}{\partial u_1 \partial u_3}; \quad E_3 = \frac{j\omega\mu}{h_2}\frac{\partial \Pi_m}{\partial u_2}.
\end{aligned} \tag{7}$$

As can be seen from (7) and (8), E-type waves have no H-component in the u_1-direction, whereas H-type waves have no E-component in that direction.
By choosing appropriate values for h_2 and h_3, expressions for the field components in Cartesian, cylindrical (including parabolic and elliptic) and even spherical coordinate systems can be obtained.
The more general case with three arbitrary scale factors gives rise to an insoluble set of interdependent equations [1, pp. 50-51].

2.4 Hertz's Wave Equation in a Cartesian Coordinate System

The three scale factors h_1, h_2 and h_3 all equal one in a right-hand Cartesian coordinate system (u_1, u_2, u_3). Hence, Hertz's vector wave equation (4) simplifies to

$$\frac{\partial^2 \Pi}{\partial u_1^2} + \frac{\partial^2 \Pi}{\partial u_2^2} + \frac{\partial^2 \Pi}{\partial u_3^2} + k^2 \Pi = 0 \tag{8}$$

with $\vec{\Pi} = \Pi(u_1, u_2, u_3)\vec{e}_1$. (9)

$\vec{e}_1$ is the unit vector in the u_1-direction.

Therefore, the field components of the E-type waves are

$$\begin{aligned} E_1 &= k^2 \Pi_e + \frac{\partial^2 \Pi_e}{\partial u_1^2}; \quad H_1 = 0, \\ E_2 &= \frac{\partial^2 \Pi_e}{\partial u_1 \partial u_2}; \quad H_2 = (\sigma + j\omega\varepsilon)\frac{\partial \Pi_e}{\partial u_3}, \\ E_3 &= \frac{\partial^2 \Pi_e}{\partial u_1 \partial u_3}; \quad H_3 = -(\sigma + j\omega\varepsilon)\frac{\partial \Pi_e}{\partial u_2}. \end{aligned} \tag{10}$$

The field components of the H-type waves are

$$\begin{aligned} H_1 &= k^2 \Pi_m + \frac{\partial^2 \Pi_m}{\partial u_1^2}; \quad E_1 = 0, \\ H_2 &= \frac{\partial^2 \Pi_m}{\partial u_1 \partial u_2}; \quad E_2 = -j\omega\mu \frac{\partial \Pi_m}{\partial u_3}, \\ H_3 &= \frac{\partial^2 \Pi_m}{\partial u_1 \partial u_3}; \quad E_3 = j\omega\mu \frac{\partial \Pi_m}{\partial u_2}. \end{aligned} \tag{11}$$

2.5 Hertz's Wave Equation for a 2D-Uniform Guiding Structure

Propagation along a guiding structure occurs in one direction only. In this text, the x-axis is chosen to be parallel with the propagation direction. Therefore, the waves along a uniform guiding structure have only an $e^{j(\omega t-\beta_x x)}$-dependence in that direction. This means that Hertz vector potentials for two-dimensional uniform guiding structures are of the form [3, p. 800]

$$\vec{\Pi} = F(y,z)e^{-j\beta_x x}\vec{e}_1$$

where $\vec{e}_1$ can be either in the x-, y- or z-direction

and phase constant $\beta_x = \beta_x' - j\beta_x'' = \frac{\gamma_x}{j}$.

The propagation of electromagnetic waves is usually characterized in terms of the propagation constant $\gamma = \alpha + j\beta$, where α is called the attenuation constant.

The formulation for β used in this text, is consistent with the expression for γ, namely $\gamma = j(\beta' - j\beta'') = \beta'' + j\beta' \;\Rightarrow\; \beta'' \equiv \alpha$.

There exist a large number of 2D-uniform guiding structures, some of the best known examples are: the parallel wire line, coaxial cable, waveguide, strip line, microstrip line, slot line and the coplanar line.

Since $\frac{\partial^2 \Pi}{\partial x^2} = -\beta_x^2 \Pi$, Hertz's scalar wave equation (8) becomes

$$\frac{\partial^2 \Pi}{\partial y^2} + \frac{\partial^2 \Pi}{\partial z^2} + s^2 \Pi = 0 \qquad (12)$$

where $s^2 = k^2 - \beta_x^2 = \varepsilon\mu\omega^2 - j\omega\mu\sigma - \beta_x^2$. (13)

Solutions to (12) can readily be found by separation of the variables. Namely, let $\Pi = Y(y)Z(z)e^{-j\beta_x x}$. (14)

Substituting (14) into (12) and dividing by (14) gives

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + s^2 = 0.$$

Since the last term in the above equation is independent of both y and z, the first two terms need to be this as well.
Therefore,

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -s_y^2; \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = -s_z^2 \quad \text{and} \quad s_y^2 + s_z^2 = s^2. \qquad (15)$$

The first two equations in (15) are linear homogeneous second order differential equations

$$\frac{d^2Y}{dy^2} + s_y^2 Y = 0; \quad \frac{d^2Z}{dz^2} + s_z^2 Z = 0.$$

Hence, suitable Hertz potential solutions for two-dimensional uniform guiding structures are of the form [6, p. 105]

$$\Pi = \left[c_1 e^{+js_y y} + c_2 e^{-js_y y}\right] \cdot \left[c_3 e^{+js_z z} + c_4 e^{-js_z z}\right] \cdot e^{-j\beta_x x}, \text{ or equally,}$$

$$\Pi = \left[c_5 \cos(s_y y) + c_6 \sin(s_y y)\right] \cdot \left[c_7 \cos(s_z z) + c_8 \sin(s_z z)\right] \cdot e^{-j\beta_x x}.$$

2.6 Hertz's Wave Equation in a Circular Cylindrical Coordinate System

In a cylindrical coordinate system, the scale factors are generally different from one, except for the scale factor associated with the symmetry axis, usually called the z-axis. In order to apply expression (4), the scale factor h_1 should equal one. Therefore, let $u_1 = z$.

The special case of a right-hand *circular* cylindrical coordinate system (r, ϕ, z) gives $u_1 = z$; $u_2 = r$ and $u_3 = \phi$. (16)

The differential line element $d\ell$ in a circular cylindrical coordinate system (r, ϕ, z) is [7]

$$d\ell = \sqrt{dr^2 + r^2 d\phi^2 + dz^2}\ .$$

The scale factors are hence [6, p. 124]

$$h_1 = \left|\frac{\partial \ell}{\partial z}\right| = 1; \quad h_2 = \left|\frac{\partial \ell}{\partial r}\right| = 1 \text{ and } h_3 = \left|\frac{\partial \ell}{\partial \phi}\right| = r\ . \tag{17}$$

Substitute (16) and (17) into (4) to get

$$\frac{\partial^2 \Pi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial \Pi}{\partial \phi}\right) + k^2\,\Pi = 0 \tag{18}$$

with $\vec{\Pi} = \Pi(z, r, \phi)\vec{e}_z$. (19)

Propagation in cylindrical symmetric transmission lines occurs in one direction only, which is usually along the z-axis. This means that the expression for the Hertz vector potentials simplifies to

$$\vec{\Pi} = F(r, \phi)e^{-j\beta_z z}\vec{e}_z\ .$$

Since $\dfrac{\partial^2 \Pi}{\partial z} = -\beta_z^2\,\Pi$, Hertz's scalar wave equation (18) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial \Pi}{\partial \phi}\right) + s^2\,\Pi = 0 \tag{20}$$

where $s^2 = k^2 - \beta_z^2 = \varepsilon\mu\omega^2 - j\omega\mu\sigma - \beta_z^2$. (21)

Solutions to (20) can readily be found by separation of the variables. Namely, let $\Pi = R(r)\Phi(\phi)e^{-j\beta_z z}$. (22)

Substituting (22) into (20) and dividing by (22) results in [3, p. 739]

$$\frac{1}{R}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + \frac{1}{\Phi}\left[\frac{1}{r}\frac{d}{d\phi}\left(\frac{1}{r}\frac{d\Phi}{d\phi}\right)\right] + s^2 = 0\ . \tag{23}$$

Multiplying (23) by r^2 gives

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+s^2r^2=0. \tag{24}$$

Equation (24) can be separated using a separation constant n into

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2, \tag{25}$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+s_r^2r^2=n^2 \tag{26}$$

where $s_r^2=s^2=k^2-\beta_z^2$.

Equation (25) is a linear homogeneous second order differential equation

$$\frac{d^2\Phi}{d\phi^2}+n^2\Phi=0.$$

Solutions for Φ are of the form [6, p. 105]

$$\Phi=c_1e^{+jn\phi}+c_2e^{-jn\phi}\text{, or equally,} \tag{27a}$$

$$\Phi=c_3\cos(n\phi)+c_4\sin(n\phi). \tag{27b}$$

Rewriting equation (26) results in an expression which can be recognized as Bessel's equation of order n [6, p. 106]

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+s_r^2r^2-n^2=0$$

$$\Rightarrow\ \frac{r}{R}\left(r\frac{d^2R}{dr^2}+1\cdot\frac{dR}{dr}\right)+s_r^2r^2-n^2=0$$

$$\Rightarrow\ r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left(s_r^2r^2-n^2\right)R=0 \tag{28}$$

with $n\geq 0$.

Solutions to Bessel's equation of order n (28) are of the form [6, p. 106], [8, pp. 97-88]

$$R=c_5J_n(s_rr)+c_6Y_n(s_rr)\text{, or equally,} \tag{29a}$$

$$R=c_7H_n^{(1)}(s_rr)+c_8H_n^{(2)}(s_rr). \tag{29b}$$

These solutions are linearly independent only if n is a positive integer.

At this point, Hertz's scalar wave equation for circular cylindrical coordinate systems (20) is solved. It suffices to substitute any form of (27) and (29) into (22) to obtain the Hertz potential solutions.

Substituting (16) and (22) into (6) gives the field components of the E-type waves expressed in terms of a Hertz potential [3, p.740]

$$E_z = s_r^2\,\Pi_e\,; \quad H_z = 0\,,$$

$$E_r = -j\beta_z \frac{\partial \Pi_e}{\partial r}\,; \quad H_r = \frac{(\sigma + j\omega\varepsilon)}{r}\frac{\partial \Pi_e}{\partial \phi}\,, \tag{30}$$

$$E_\phi = -j\frac{\beta_z}{r}\frac{\partial \Pi_e}{\partial \phi}\,; \quad H_\phi = -(\sigma + j\omega\varepsilon)\frac{\partial \Pi_e}{\partial r}\,.$$

Likewise, substitute (16) and (22) into (7) to obtain the field components of the H-type waves

$$H_z = s_r^2\,\Pi_m\,; \quad E_z = 0\,,$$

$$H_r = -j\beta_z \frac{\partial \Pi_m}{\partial r}\,; \quad E_r = -j\frac{\omega\mu}{r}\frac{\partial \Pi_m}{\partial \phi}\,, \tag{31}$$

$$H_\phi = -j\frac{\beta_z}{r}\frac{\partial \Pi_m}{\partial \phi}\,; \quad E_\phi = j\omega\mu\frac{\partial \Pi_m}{\partial r}\,.$$

2.7 Conclusions

Chapter 2 gave a review of Hertz potential theory. The convenience of expressing source free electromagnetic fields in terms of Hertz potentials was clearly demonstrated. The theory is kept as general as possible, making it useful as a reference while solving many other electromagnetic problems.

2.8 References

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